Time-Domain Transmission Matrix of Lossy Transmission Lines

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***Abstract*—In this letter the time-domain transmission matrix T of lossy transmission lines is numerically derived by the method of characteristics without employing any convolution and Fourier transform, and some interesting properties about the matrix are given. The transmission lines can be uniform or nonuniform. Transmission line parameters are explicit in the matrix, which makes it very useful in some practical applications.**

***Index Terms*—Characteristics, transmission line, transmission matrix.**

I. Introduction

TERMINAL response voltages and currents of transmission lines are most conveniently related through the frequency-domain network parameters, such as the $\boldsymbol{Y}$ parameters and transmission parameters. In order to get the time-domain response relationship of lossy transmission lines, convolution of the time-domain Green's functions (which are usually obtained by Fourier transforms of the frequency-domain network parameters) with the terminal responses has to be performed. The transmission line parameters (R, C, L, and G) are not explicit in the expressions, which seriously restricts the application of such time-domain relationship. Moreover, the Fourier transform and convolution suffer from efficiency and accuracy problems.

In this letter, a simple numerical expression of the time-domain transmission matrix $\boldsymbol{T}$ of a lossy transmission line is obtained with the method of characteristics [1], [2], without using convolution and Fourier transform. The transmission line can be nonuniform. Through the time-domain transmission matrix $\boldsymbol{T}$, sample values of the near-end response voltage and current can be related with sample values of the far-end response voltage and current. The transmission matrix $\boldsymbol{T}$ is the product of m sub-transmission matrix $\boldsymbol{T}_k(k = 1, 2, \cdots, m)$, where m is the number of uniform sections which together approximate a nonuniform line. All the subtransmission matrices $\boldsymbol{T}_k$ have the same structure and are very sparse. Some interesting properties and the relevant physical meaning of the transmission matrix $\boldsymbol{T}$ are given.

The elements of $\boldsymbol{T}_k$ are rational expressions of R_k, C_k, L_k, and G_k of the kth line section, so the transmission line parameters are explicit in the time-domain transmission matrix $\boldsymbol{T}$. This property makes $\boldsymbol{T}$ very useful in some practical applications such as pulse shaping and transmission line parameter extraction. In this letter we have designed a lossy tapered transmission line which has distortionless far-end response before the reflected waves from the near-end arrive at the far-end.

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II. Time-Domain Transmission Matrix $\boldsymbol{T}$

In this letter a nonuniform transmission line of parameters $R(x), C(x), L(x), G(x)$ and length D is represented by m short sections of uniform lines with parameters R_k, C_k, L_k, and $G_k(k = 1, 2, \cdots, m)$. For convenience we assume the uniform sections may be of different lengths but of the same time delay $h = \tau/m$, where τ is the time delay of the original nonuniform line defined by

$$\tau = \int_0^D \sqrt{L(x)C(x)}\, dx. \tag{1}$$

Let x_k and $x_{k+1}(k = 1, 2, \cdots, m)$ be the terminal longitudinal coordinates of the kth uniform section and $v_{k,j}$ and $i_{k,j}$ be the sample values of the response voltage $v(x, t)$ and current $i(x, t)$ at $x = x_k$ and $t = jh$, $j = 0, 1, 2, \cdots$. (See Fig. 1.) The characteristic curves of the nonuniform line represented by m uniform sections are piecewise straight lines with slopes of $(L_k C_k)^{0.5}$, $k = 1, 2, \cdots, m$. In [3], we have proven that at any point under the dark characteristic curve OSJH ... TP the responses are of *zero* values, and that along OSJH ... TP there is

$$v(x,t) = Z(x)i(x,t), \qquad x < D \tag{2}$$

where $Z(x)$ is the characteristic impedance of the transmission line at position x, which is defined by

$$Z(x) = \sqrt{\frac{L(x^+)}{C(x^+)}} \tag{3}$$

where x^+ denotes $x + \Delta x$ when Δx tends to *zero*, but is larger than *zero*. According to the Trapezoidal algorithm of the numerical characteristics method [2], we have

$$\begin{aligned}(1-\Phi_k)v_{k,j} &+ (Z_k - \Psi_k)i_{k,j}\\ &= (1+\Phi_k)v_{k+1,j+1} + (Z_k + \Psi_k)i_{k+1,j+1} \end{aligned}\tag{4.1}$$

$$\begin{aligned}(1+\Phi_k)v_{k,j} &- (Z_k + \Psi_k)i_{k,j}\\ &= (1-\Phi_k)v_{k+1,j-1} - (Z_k - \Psi_k)i_{k+1,j-1}\\ &\qquad k = 1, 2, \cdots, m, j > k \end{aligned}\tag{4.2}$$

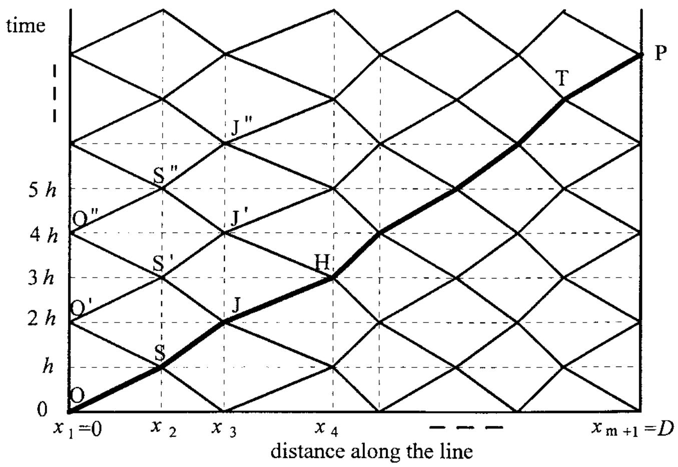

Fig. 1. The piecewise straight characteristic curves of a nonuniform transmission line represented by m uniform sections.

in which

$$Z_k = \sqrt{\frac{L_k}{C_k}}, \quad \Psi_k = \frac{0.5hR_k}{\sqrt{L_kC_k}}, \quad \Phi_k = \frac{0.5hG_k}{C_k} \tag{5}$$

are the characteristic impedance, one half of the total resistance, and one half of the multiplying product of Z_k with the total conductance of the kth uniform section, respectively. From (4.1), (4.2), and (2) we get

$$\begin{aligned} v_{k,j} =& a_k v_{k+1,j+1} + b_k v_{k+1,j-1} + c_k i_{k+1,j+1} \\ &+ d_k i_{k+1,j-1} \\ i_{k,j} =& e_k v_{k+1,j+1} + f_k v_{k+1,j-1} + a_k i_{k+1,j+1} \\ &+ b_k i_{k+1,j-1} \end{aligned} \tag{6}$$

and

$$\begin{aligned} v_{k,j} &= p_k v_{k+1,j+1} + q_k i_{k+1,j+1} \\ i_{k,j} &= r_k v_{k+1,j+1} + s_k i_{k+1,j+1}, \quad k = j+1, 2, \cdots, m \end{aligned} \tag{7}$$

where

$$a_k = \frac{(Z_k + \Psi_k)(1 + \Phi_k)}{2(Z_k - \Phi_k \Psi_k)}, \quad b_k = \frac{(Z_k - \Psi_k)(1 - \Phi_k)}{2(Z_k - \Phi_k \Psi_k)} \tag{8}$$

$$c_k = \frac{(Z_k + \Psi_k)(Z_k + \Phi_k)}{2(Z_k - \Phi_k \Psi_k)}, \quad d_k = \frac{(Z_k - \Psi_k)(Z_k - \Phi_k)}{2(Z_k - \Phi_k \Psi_k)} \tag{9}$$

$$e_k = \frac{(1 + \Psi_k)(1 + \Phi_k)}{2(Z_k - \Phi_k \Psi_k)}, \quad f_k = \frac{(1 - \Psi_k)(1 - \Phi_k)}{2(Z_k - \Phi_k \Psi_k)} \tag{10}$$

$$p_k = \frac{Z_k(1 + \Phi_k)}{2Z_k - Z_k\Phi_k - \Psi_k}, \quad q_k = \frac{Z_k(Z_k + \Psi_k)}{2Z_k - Z_k\Phi_k - \Psi_k} \tag{11}$$

$$r_k = \frac{1 + \Phi_k}{2Z_k - Z_k\Phi_k - \Psi_k}, \quad s_k = \frac{Z_k + \Psi_k)}{2Z_k - Z_k\Phi_k - \Psi_k}. \tag{12}$$

From (6) and (7) we can derive the relationship between the response sample values at two ends of the kth uniform section:

$$\begin{pmatrix} \boldsymbol{v}_k \\ \boldsymbol{i}_k \end{pmatrix} = \boldsymbol{T}_k \begin{pmatrix} \boldsymbol{v}_{k+1} \\ \boldsymbol{i}_{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{T1}_k & \boldsymbol{T2}_k \\ \boldsymbol{T3}_k & \boldsymbol{T4}_k \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_{k+1} \\ \boldsymbol{i}_{k+1} \end{pmatrix}, \quad k = 1, 2, \cdots, m \tag{13}$$

in which

$$\boldsymbol{v}_k = (v_{k,k-1}, v_{k,k+1}, \cdots, v_{k,k+2n-3})^T \tag{14}$$

$$\boldsymbol{i}_k = (i_{k,k-1}, i_{k,k+1}, \cdots, i_{k,k+2n-3})^T \tag{15}$$

are column vectors of voltage and current sample values of dimension n, and n is an arbitrary positive integer. In (14) and (15), the superscript "T" denotes transposition. In (13) $\boldsymbol{T1}_k, \boldsymbol{T2}_k, \boldsymbol{T3}_k$, and $\boldsymbol{T4}_k$ are square matrices of dimension n with elements given by

$$\left.\begin{aligned} \boldsymbol{T1}_k(i,j) &= p_k \\ \boldsymbol{T2}_k(i,j) &= q_k \\ \boldsymbol{T3}_k(i,j) &= r_k \\ \boldsymbol{T4}_k(i,j) &= s_k \end{aligned}\right\}, \quad \text{for } i = j = 1 \tag{16}$$

$$\left.\begin{aligned} \boldsymbol{T1}_k(i,j) &= a_k \\ \boldsymbol{T2}_k(i,j) &= c_k \\ \boldsymbol{T3}_k(i,j) &= e_k \\ \boldsymbol{T4}_k(i,j) &= a_k \end{aligned}\right\}, \quad \text{for } i < i = j \le n \tag{17}$$

$$\left.\begin{aligned} \boldsymbol{T1}_k(i,j) &= b_k \\ \boldsymbol{T2}_k(i,j) &= d_k \\ \boldsymbol{T3}_k(i,j) &= f_k \\ \boldsymbol{T4}_k(i,j) &= b_k \end{aligned}\right\}, \quad \text{for } 1 < i = j + 1 \le n \tag{18}$$

$$\left.\begin{aligned} \boldsymbol{T1}_k(i,j) &= 0 \\ \boldsymbol{T2}_k(i,j) &= 0 \\ \boldsymbol{T3}_k(i,j) &= 0 \\ \boldsymbol{T4}_k(i,j) &= 0 \end{aligned}\right\}, \quad \text{for other } 1 \le i \le n, 1 \le j \le n. \tag{19}$$

In (16)–(19), $\boldsymbol{T1}_k(i,j), \boldsymbol{T2}_k(i,j), \boldsymbol{T3}_k(i,j)$, and $\boldsymbol{T4}_k(i,j)$ are elements at the ith row and jth column of the matrices $\boldsymbol{T1}_k, \boldsymbol{T}_k, \boldsymbol{T}_k$, and $\boldsymbol{T4}_k$, respectively. Repeatedly using (13) from $k = 1$ to $k = m$, we get the time-domain relationship between the terminal responses of the original nonuniform transmission line:

$$\begin{pmatrix} \boldsymbol{v}_n \\ \boldsymbol{i}_n \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} \boldsymbol{v}_f \\ \boldsymbol{i}_f \end{pmatrix} = \begin{pmatrix} \boldsymbol{T1} & \boldsymbol{T2} \\ \boldsymbol{T3} & \boldsymbol{T4} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}_f \\ \boldsymbol{i}_f \end{pmatrix} \tag{20}$$

in which $\boldsymbol{v}_n$ and $\boldsymbol{i}_n$ are vectors of the near-end voltage and current sample values at time $t = 2jh, j = 0, 1, \cdots, n-1$, $\boldsymbol{v}_f$ and $\boldsymbol{i}_f$ are vectors of the far-end voltage and current sample values at time $t = 2jh + \tau$, $j = 0, 1, \cdots, n-1$, and

$$\boldsymbol{T} = \prod_{k=1}^{m} \boldsymbol{T}_k \tag{21}$$

is the time-domain transmission matrix. $\boldsymbol{T1}$–$\boldsymbol{T4}$ are four square partition matrices of dimension n of matrix $\boldsymbol{T}$. From the element expressions (16)–(19), several properties about the transmission matrix $\boldsymbol{T}$ and its partition matrices can be obtained as in the following.

1) The partition matrices $\boldsymbol{T1}_k$–$\boldsymbol{T4}_k$ are very sparse with *nonzero* elements only at the diagonal and lower subdiagonal positions.
2) The transmission matrix $\boldsymbol{T}$ is singular and has no inverse matrix. The physical meaning of this property is that the far-end responses before time $\tau + t$ can not be derived from the near-end responses before time t, although the latter can be derived from the former according to (20). In fact, the far-end responses before time $\tau + t$ are decided by both the transmission line parameters and the

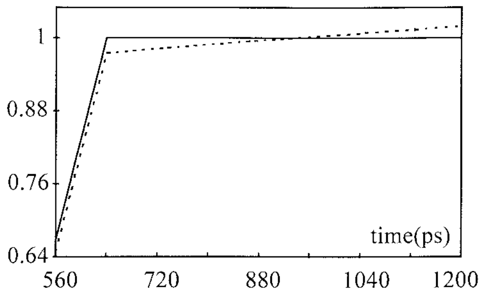

Fig. 2. The far-end response voltage in the example. – – –: distorted response of the lossy uniform line; ——: distortionless response of the tapered line.

boundary conditions at the near-end and far-end, while the near-end responses before time $t < 2\tau$ includes no information of the boundary condition at the far-end.

3) In the special case of uniform transmission lines, along the characteristic line $t = (LC)^{0.5}x$ we have

$$v(0,0) = 0.5e^{0.5((G/C)+(R/L))\tau}(v(D,\tau) + Zi(D,\tau)) \tag{22}$$

where Z is the characteristic impedance of the uniform line.

III. Application Example

In this section we use (20) to design a lossy tapered transision line which has distortionless far-end response. Consider a uniform transmission line with conductor width $w = 30$ mm, conductor thickness 5 mm and dielectric height 90 mm. The dielectric constant is $\epsilon_r = 9.8$. The line is excited by a pulse signal of 240-ps rise time and connected to 40 and 450 Ω at the source end and another end, respectively. Using the computed Z and R parameters from the line sizes by the empirical formula in [4], we get the far-end response voltage shown in Fig. 2 as the dashed curve in the time range $560 \text{ ps} < t < 3\tau = 1200$ ps, where τ is the time delay of the transmission line. Obvious distortion is observed in the response which is induced by the line loss. To reduce the distortion, we can taper the uniform line to a nonuniform one which is composed of $m = \tau/h$ uniform sections, so that the tapered line has sample far-end response values of γe_j at time points $t = \tau + 2jh$, $j = 0, 1, \cdots, m-1$, where e_j are the sample values of the source signal at time point $t = 2jh$, $j = 0, 1, \cdots, m-1$, γ is a real positive number to be decided. In practical cases of fabricating transmission lines, the only parameter which can be adjusted is the conductor width, so the characteristic impedance Z_k and the resistance parameter Ψ_k of the uniform sections are not independent variables and can be solved from (20). Then, the conductor widths of the uniform sections can be derived from impedance-width curves of quasi-TEM analysis or ful-wave analysis. The obtained conductor widths, characteristic impedances, and halves of resistances of the uniform sections are all given in Fig. 3. The linear factor is found to be $\gamma = 0.9996$. The far-end response voltage of such a tapered line is also computed by the characteristics method and is shown in Fig. 2 as the solid curve between $t = 560$ ps and $t = 3\tau$. Due to the appropriate selection of terminal loads this response signal is not only distortionless, but also with the same amplitude as the exciting signal.

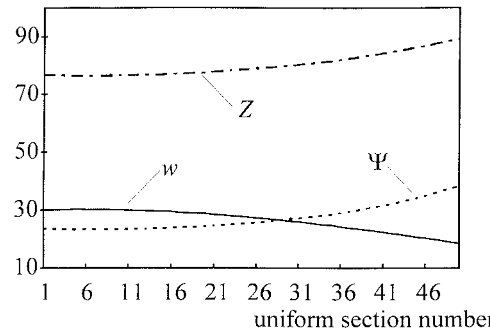

Fig. 3. ——: the width (W) in mm; — - —: the characteristic impedance (Z) in Ω; – – –: the resistance (Ψ) in $10^2\Omega$ of the uniform sections which represent the tapered line.

References

[1] N. Orhanovic, V. K. Tripathi, and P. Wang, "Time domain simulation of uniform and nonuniform multiconductor lossy lines by the method of characteristics," in *Proc. MTT-S*, May 1990, pp. 1191–1194.

[2] J.-F. Mao and E. S. Kuh, "Fast simulation and sensitivity analysis of lossy transmission lines by the method of characteristics," *IEEE Trans. Circuits Syst. I,* vol. 44, pp. 391–401, May 1997.

[3] J.-F. Mao, O. Wing, and F.-Y. Chang, "Transmission line synthesis by the method of characteristics," *IEEE Trans. Circuit Syst. I,* vol. 43, pp. 461–468, June 1996.

[4] H. T. Youn, Y. Lin, and S. Y. Chiang, "Properties of interconnection on silicon, sapphire, and semiinsulating gallium arsenide substrates," *IEEE Trans. Electron. Devices,* p. 439, Apr. 1982.